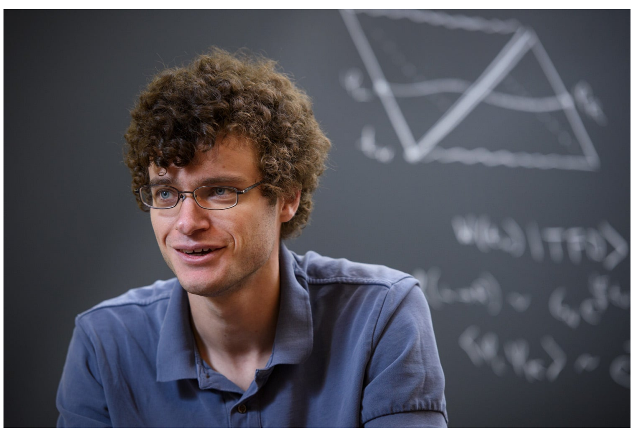
Vacuum sphere and FZZT disk partition functions in minimal string theory

2107.01172 with Douglas Stanford & Cynthia Yan



See also
2106.01665 by Anninos, Bautista and Muhlmann
2106.04532 by Muhlmann.

In string theory, the sphere partition function without operator insertions is a fundamental but confusing quantity. There is an expectation that it should give minus the classical value of the on-shell action of the string background. There are backgrounds of ordinary critical string theory where the on-shell action is nonzero and physically important, like for the $\mathrm{AdS}_3 \times S^3 \times X$ background.

Part of the puzzle is that one has to divide by the volume of the conformal Killing group $\mathrm{PSL}(2, \mathbb{C})$, which has infinite volume and no sensible finite regularized value. Another aspect is that in cases where the partition function is expected to be nonzero, the target space is noncompact, and there is a divergent integral over the location of the string worldsheet.

This is unlike the disk case where Liu and Polchinski showed that $\mathrm{PSL}(2, \mathbb{R})$ has a finite regularized value. The supersymmetric version of $\mathrm{PSL}(2, \mathbb{R})$ is $\mathrm{OSp}(1|2, \mathbb{R})$ and that group has finite volume. One can use this to give a direct path integral computation of the D-brane tension (in terms of α' and κ, the 10d Newton's constant).

What seems reasonably clear is that the sphere diagram is zero up to effects having to do with the noncompactness of the target space. This is consistent with the fact that in the low-energy expansion, the on-shell action of string theory vanishes up to boundary terms [Tsyetlin].

To compute the on-shell action of a noncompact spacetime, one has to put some kind of radial cutoff, add the GHY boundary term together with additional counterterms, and take a limit.

As a possibly irrelevant warmup, we compute the sphere partition function in noncritical string theory where the answer is definite and can be checked against a matrix integral computation. We study the noncritical string theory consisting of Liouville theory and the $(2, p)$ minimal model, and we do the Liouville path integral directly in the semiclassical limit $c_{\text{Liouville}} \to +\infty$, which is relevant for large values of p.

For the conceptual points we are interested in (and for controlling the overall normalization) the direct semiclassical calculation is useful.

This computation was previously done by Zamolodchikov [1], but we go a little further and compute the one-loop determinant.

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ON THE ENTROPY OF RANDOM SURFACES

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The main conceptual point is that in this example, *the conformal symmetry of the Liouville theory is spontaneously broken by a saddle point of the Liouville field. The integral over the noncompact Goldstone bosons cancels against the divergent volume of* $PSL(2,\mathbb{C})$ *that we are supposed to divide by.*

However, there are some further more conventional subtleties that make the sphere partition function not well-defined. The ratio sphere/disk^2 is well defined though, and we compute this and match it to matrix integral predictions, including precise order one multiplicative constants.

1 Liouville computations

We will compute Liouville path integrals on the sphere, and on the disk (hemisphere) with FZZT boundary conditions, in a semiclassical approximation at large positive Liouville central charge.

There are three components to this calculation:

1. Identifying the relevant saddles and the steepest descent contours
2. Computing the one-loop determinants
3. Dividing by the volume of the gauge group

A first point is the existence of noncompact manifold of saddle points. The origin of these zero modes is that the saddle point configurations of the Liouville field spontaneously break the conformal symmetry, leading to three Goldstone modes for the sphere and two Goldstone modes for the disk.

On the sphere, the globally defined conformal symmetry group is $\mathrm{PSL}(2,\mathbb{C})$, and on the disk it is $\mathrm{PSL}(2,\mathbb{R})$, and the zero modes parametrize the quotient space G/H where $G = \mathrm{PSL}(2,\mathbb{C})$ or $\mathrm{PSL}(2,\mathbb{R})$ and $H = \mathrm{PSU}(2)$ or $\mathrm{U}(1)$.

A second subtlety is that the overall normalization of the path integral is ambiguous, due to

1. the conformal anomaly
2. the existence of a finite counterterm proportional to the Euler characteristic
3. an arbitrary choice of measure on the group G whose volume we divide by

These ambiguities really exist, but they can be made to cancel out in the ratio

$$\frac{Z_{\text{sphere}}}{\text{vol}(\text{PSL}(2,\mathbb{C}))} \cdot \left(\frac{\text{vol}(\text{PSL}(2,\mathbb{R}))}{Z_{\text{disk}}}\right)^2. \tag{1.1}$$

In order to make the ambiguities (1) and (2) cancel out, we will use the same metric for the two problems, taking the disk to the be the hemisphere. We will also use the same cutoff procedure for the computation of the sphere and the disk one loop determinants.

To determine the relative gauge group measures on the sphere and the disk, we note that in non-critical string theory, the factors of $1/\mathrm{vol}(G)$ arise from zero mode integrals in the path integral over the bc ghosts. Thus the measure on the gauge group arises from the normalization of the c-ghost zero modes on the sphere and the hemisphere.

Define the Liouville field σ so that the physical metric is

$$\mathrm{d}s^2 = e^{2\sigma}\mathrm{d}\hat{s}^2. \tag{1.2}$$

In the explicit computations, we will use the sphere or hemisphere as the reference metric

$$\mathrm{d}\hat{s}^2 = d\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2, \quad \hat{R} = 2, \quad \hat{K}|_{\mathrm{equator}} = 0.$$

It is conventional to write the central charge of Liouville theory as $c = 1 + 6(1/b + b)^2$, and to approach the limit of large c by taking b small.

The Liouville action is

$$I = \frac{1}{b^2}\left\{\frac{1}{4\pi}\int\sqrt{\hat{g}}\left[(\partial\sigma)^2 + \hat{R}\sigma + 4\pi\mu e^{2\sigma}\right]\right.$$

$$\left. + \frac{1}{2\pi}\int\sqrt{\hat{h}}\left[\hat{K} + 2\pi\mu_B e^{\sigma}\right]\right\} \tag{1.3}$$

$$+ \frac{1}{4\pi}\int\sqrt{\hat{g}}\hat{R}\sigma + \frac{1}{2\pi}\int\sqrt{\hat{h}}\hat{K}\sigma. \tag{1.4}$$

The parameter μ is called the cosmological constant, and the parameter μ_B is called the boundary cosmological constant. Our conventions for these parameters differ by a factor of b^2 from the standard ones in the literature.

1.1 Classical solutions and action

1.1.1 Sphere

On the sphere, there is a simple family of classical solutions given by constant configurations of σ. Restricting to such configurations, the equation of motion is

$$\begin{aligned} & 2 + 8\pi\mu e^{2\sigma} = 0 \\ \implies \quad & 2\sigma = \log(\frac{1}{4\pi\mu}) + \mathrm{i}\pi(1+2n). \end{aligned}$$

We see that there are actually an integer-indexed family of solutions, in which the Liouville field σ differs by $2\pi \mathrm{i} n$.

Plugging the solutions into the action, we find

$$e^{-I_{\text{classical}}} = -e^{-\frac{\mathrm{i}\pi}{b^2}(1+2n)} e^{\frac{1}{b^2}} (4\pi\mu)^{\frac{1}{b^2}+1}.$$

One might be surprised by the fact that there are any classical solutions for Liouville theory on a spherical topology, given that the equations of motion impose that the physical metric $e^{2\sigma}\mathrm{d}\hat{s}^2$ should have constant negative curvature, and that no everywhere-negative-curvature metric is possible on a spherical topology.

In fact, for the solutions we described, the metric $e^{2\sigma}\mathrm{d}\hat{s}^2$ is a round sphere with an overall negative sign in front. Formally, these solutions have negative curvature $R < 0$ and count as valid complex solutions to the equations of motion (see related discussion in [2] in JT gravity).

Which, if any, of these solutions are we supposed to sum over? If the theory is defined by analytic continuation in b, starting from the region where b has a positive imaginary part, then the correct answer is to sum the solutions with $n = 0, 1, 2, \ldots$ [3]. This gives the result

$$\sum_{n=0}^{\infty} e^{-I_{\text{classical}}} = \frac{\mathrm{i}}{2\sin(\frac{\pi}{b^2})} e^{\frac{1}{b^2}} (4\pi\mu)^{\frac{1}{b^2}+1}. \quad (1.5)$$

In principle, one should sum over the saddle points at the end, after including the one-loop determinants and gauge fixing factors, but these are independent of n, so it is allowable to sum over the saddles at this early stage.

To motivate this prescription, one can consider a toy integral

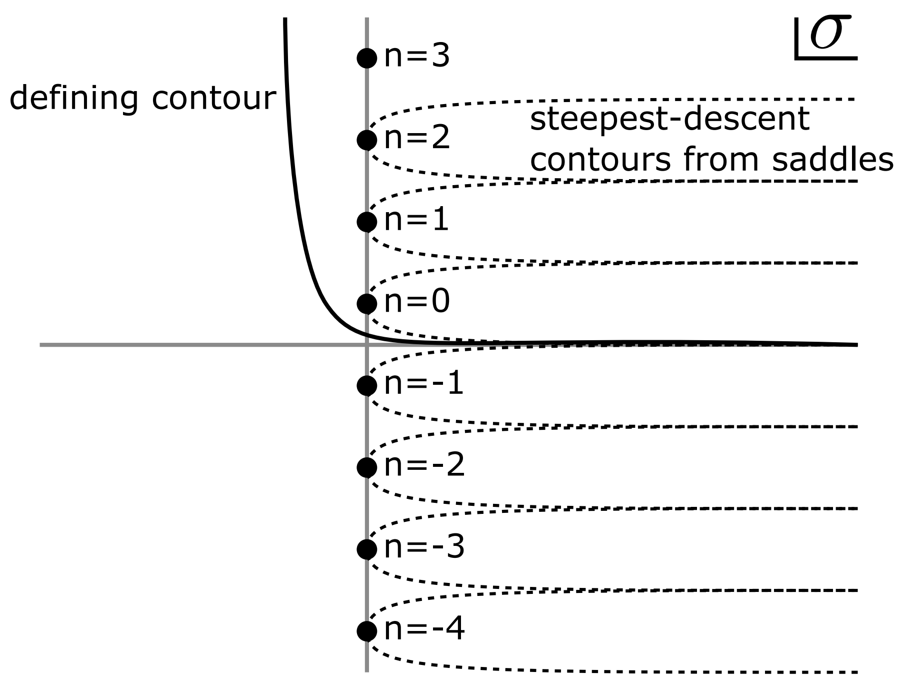
$$\int_{-\infty}^{\infty} \mathrm{d}\sigma \, e^{a\sigma - e^{2\sigma}} \tag{1.6}$$

After setting $a = -2(b^{-2} + 1)$ and shifting σ by a constant, this corresponds to the truncation of the Liouville path integral to the constant mode of σ.

If the real part of a is positive, then the integral converges along the real axis.

But for our problem, a is negative, and the integral does not converge, but we can imagine defining it by analytic continuation.

If we vary a through the upper half plane from positive values almost all the way to the negative real axis, then one acceptable defining contour is the one shown below:



(1.7)

1.1.2 FZZT disk

On the disk (hemisphere), a constant σ is not a solution to the equations of motion. The next simplest thing is to find solutions $\sigma(\theta)$ that are independent of the angular coordinate ϕ.

The equations of motion are

$$1 + 4\pi\mu e^{2\sigma} = \frac{1}{\sin(\theta)}(\sigma' \sin(\theta))' \tag{1.8}$$

$$2\pi\mu_B e^{\sigma(\pi/2)} + \sigma'(\pi/2) = 0 \tag{1.9}$$

One can check that the following is a solution

$$\begin{aligned} \sigma &= 2\pi i n - \frac{1}{2}\log(4\pi\mu) \\ &\qquad + \log\left[\frac{2\alpha}{(1+\alpha^2)\cos(\theta) + (1-\alpha^2)}\right], \\ 1 &+ \alpha^2 + 2\sqrt{\frac{\pi}{\mu}}\mu_B\alpha = 0. \end{aligned}$$

The on-shell action is given by

$$I = \frac{1}{b^2}\left[2\pi i n - \frac{1}{2} + \log(\alpha) - \frac{1}{2}\log(4\pi\mu)\right] + \left[-\frac{1}{2}\log(4\pi\mu) + 1 + \frac{2\log\alpha - (1-\alpha^2)\log\frac{2\alpha}{1-\alpha^2}}{1+\alpha^2}\right].$$

When we compute the one-loop determinants later, we will actually only do the computation in the limit of small positive α, which corresponds to large negative μ_B. Physically, this is a high energy limit in the matrix integral. In this limit, we have the leading behavior

$$e^{-I_{\mathrm{classical}}} = \frac{2}{e} e^{-2\pi i n/b^2} e^{\frac{1}{2b^2}} \left(\frac{\sqrt{4\pi\mu}}{\alpha}\right)^{\frac{1}{b^2}+1}.$$

Again, there is an integer-indexed family of solutions, and one has to decide which solutions should be included. We will assume that it is correct to imitate the case of the sphere, and sum over $n = 0, 1, 2, \ldots$, which leads to the answer

$$\sum_{n=0}^{\infty} e^{-I_{\text{classical}}} = -\frac{\mathrm{i}e^{\mathrm{i}\pi/b^2}}{2\sin(\frac{\pi}{b^2})} \cdot \frac{2}{e} e^{\frac{1}{2b^2}} \left(\frac{\sqrt{4\pi\mu}}{\alpha}\right)^{\frac{1}{b^2}+1}$$

Our understanding of the contour is not as good for this case as for the sphere, but one piece of evidence for this formula is that exact Liouville formulas [4, 5] do contain a factor of $1/\sin(\pi/b^2)$, which arises in this expression from the sum over saddles.

1.2 One loop determinants

To compute the one-loop determinant, we expand around a classical solution

$$\sigma = \sigma_{\mathrm{cl}} + \chi \tag{1.10}$$

and integrate over the fluctuation χ with an appropriate action and measure. The measure is derived from an ultralocal metric in field space

$$\mathrm{d}s^2 = C^2 \cdot (\mathrm{d}\chi, \mathrm{d}\chi) \tag{1.11}$$

where we introduced an arbitrary constant C to parametrize the normalization ambiguity in the metric, and we defined

$$(f, g) = \frac{1}{4\pi} \int \mathrm{d}^2 x \, \sqrt{\hat{g}} \, f(x) g(x). \tag{1.12}$$

1.2.1 Sphere

Expanding around any of the classical solutions for the sphere topology, the quadratic part of the action (1.3) is

$$I \supset \frac{1}{4\pi b^2} \int \sqrt{\hat{g}} \left[(\partial\chi)^2 - 2\chi^2 \right]. \tag{1.13}$$

The eigenfunctions of this problem are the spherical harmonics, and the eigenvalues are

$$\lambda = \ell(\ell+1) - 2, \quad \text{degeneracy } 2\ell + 1. \tag{1.14}$$

We have to deal separately with the $\ell = 0$ eigenfunction, the $\ell = 1$ eigenfunctions, and all of the others.

First, note that the $\ell = 0$ eigenfunction is a negative mode, with eigenvalue $\lambda = -2$. This is to be expected based on the diagram (1.7). In that diagram, the steepest-descent contours pass vertically through the saddle points, which means that the action is unstable with respect to real perturbations in the constant mode of χ. In the quadratic approximation, the steepest descent contour is just the imaginary axis, and the integral is

$$C \int_{+\mathrm{i}\infty}^{-\mathrm{i}\infty} \mathrm{d}\chi_0 e^{2\chi_0^2/b^2} = -\mathrm{i}\frac{\sqrt{\pi}\, bC}{\sqrt{2}}. \tag{1.15}$$

Next, the $\ell = 0$ modes are the zero modes that we promised. These correspond to the Goldstone modes of the $\mathrm{PSL}(2, \mathbb{C})$ symmetry that is spontaneously broken by the classical solutions. We will take these properly into account in the gauge-fixing part of the computation; for now we simply insert delta functions, so the contribution of these modes is

$$\prod_{i=1}^{3} C \int \mathrm{d}\chi_i \delta(\chi_i) = C^3. \tag{1.16}$$

Finally, we have the product over all of the other modes with $\ell \geq 2$:

$$\prod_{\ell=2}^{\infty} \left[\frac{\sqrt{\pi} b C}{\sqrt{\ell(\ell+1) - 2}} \right]^{2\ell+1} . \tag{1.17}$$

This is a divergent product, but we can compute a regularized version

$$\sum_{\lambda > 0} \log \left[\frac{\sqrt{\pi} b C}{\sqrt{\lambda}} \right] e^{-\epsilon^2 \lambda} =$$
$$\sum_{\ell=2}^{\infty} (2\ell + 1) \log \left[\frac{\sqrt{\pi} b C}{\sqrt{\ell(\ell+1) - 2}} \right] e^{-\epsilon^2 \ell(\ell+1)}$$

One needs to use the following sums:

$$\sum_{\ell=0}^{\infty}(2\ell+1)e^{-\epsilon^2\ell(\ell+1)} = \frac{1}{\epsilon^2} + \frac{1}{3} + O(\epsilon^2)$$

$$\sum_{\ell=2}^{\infty}(2\ell+1)\log\Big[\ell(\ell+1)-2\Big]e^{-\epsilon^2\ell(\ell+1)} =$$

$$\frac{\log(\frac{1}{\epsilon^2})-\gamma}{\epsilon^2} - 2\log(\frac{1}{\epsilon^2}) + 2.32713 + O(\epsilon^2),$$

where γ is the Euler-Mascheroni constant. We determined the divergent terms by approximating the sums as integrals, and we determined the constant terms numerically.

1.2.2 Hemisphere

Expanding around the solution with the disk topology to quadratic order and taking the limit of small α, one finds

$$I \approx \frac{1}{b^2}\left\{\frac{1}{4\pi}\int_{\text{hemisphere}}\sqrt{\hat{g}}(\partial\chi)^2 - \frac{1}{4\pi}\int_{\text{equator}}\chi^2\right\}.$$

The eigenfunctions are determined by solving

$$-\partial^2\chi = \lambda\chi, \tag{1.18}$$

with the following boundary condition at the equator

$$\chi'(\pi/2) = \chi(\pi/2). \tag{1.19}$$

The solutions are

$$P_\ell^m(\cos(\theta))e^{im\phi}, \quad \lambda = \ell(\ell+1) \tag{1.20}$$

where P is the generalized Legendre function and ℓ is a (non-integer!) parameter that is determined by solving the boundary condition equation.

The spectrum is qualitatively similar to the one-loop spectrum on the sphere. There is one negative eigenvalue, two zero modes, and there are an infinite number of other eigenvalues for which the product requires regularization.

The negative mode is in the $m = 0$ sector, and numerically, its eigenvalue is $\lambda_0 \approx -1.51095$. Its contribution to the one-loop determinant is

$$C \int_{+\mathrm{i}\infty}^{-\mathrm{i}\infty} \mathrm{d}\chi_0 e^{-\lambda_0 \chi_0^2/b^2} \approx -\mathrm{i}\frac{\sqrt{\pi}\, bC}{\sqrt{1.51095}}. \quad (1.21)$$

The two zero modes are $\chi = e^{\pm \mathrm{i}\phi} \tan\frac{\theta}{2}$. These correspond to the two spontaneously broken generators of $\mathrm{PSL}(2, \mathbb{R})$. We will treat these in the gauge-fixing step, but for now we insert delta functions, so they contribute

$$\prod_{i=1}^{2} C \int \mathrm{d}\chi_i \delta(\chi_i) = C^2. \quad (1.22)$$

Next we discuss the product over all of the other modes. We need to regularize the disk in a different way, by inserting in the sum a slightly different convergence factor $e^{-\epsilon^2\widetilde{\lambda}_i}$ where

$$\widetilde{\lambda}_i = \frac{1}{4\pi}\int \sqrt{\hat{g}}(\partial Y_i)^2 \tag{1.23}$$

$$= \lambda_i + \frac{1}{4\pi}\int_{\mathrm{bdy}} Y_i^2. \tag{1.24}$$

The upshot is that one finds the revised answers for the sums

$$\sum_{\lambda>0} e^{-\epsilon^2\widetilde{\lambda}} = \frac{1}{2\epsilon^2} + \frac{\sqrt{\pi}}{4\epsilon} - \frac{17}{6} + O(\epsilon)$$

$$\sum_{\lambda>0} \log(\lambda)e^{-\epsilon^2\widetilde{\lambda}} = \frac{\log\frac{1}{\epsilon^2} - \gamma}{2\epsilon^2} +$$

$$\sqrt{\pi}\frac{\log\frac{1}{\epsilon^2} - \log(4) - \gamma}{4\epsilon} - \log(\frac{1}{\epsilon^2})$$

$$+ 1.14858 + O(\epsilon)$$

1.3 Dividing by the volume of the conformal group

Liouville theory on the sphere has an exact $\mathrm{PSL}(2,\mathbb{C})$ conformal symmetry. An $\mathrm{PSU}(2)$ subgroup of this corresponds to ordinary rotations of the sphere; these symmetries are preserved by the classical saddle points. However, the remaining three directions in $\mathrm{PSL}(2,\mathbb{C})$ are spontaneously broken, which means that if we act with an $\mathrm{PSL}(2,\mathbb{C})$ generator in this subspace, it changes the saddle point nontrivially to a new saddle point with shifted values of χ_1, χ_2, χ_3.

Starting with the measure for the field zero modes, divided by the measure on $\mathrm{PSL}(2,\mathbb{C})$, we replace it as follows

$$\frac{\mathrm{d}(\text{sphere zero modes})}{\mathrm{d}(\mathrm{PSL}(2,\mathbb{C}))} = \frac{\mathrm{d}\chi_1\mathrm{d}\chi_2\mathrm{d}\chi_3}{\underbrace{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}_{\mathrm{PSU}(2)} \quad \underbrace{\mathrm{d}b_1\mathrm{d}b_2\mathrm{d}b_3}_{\mathrm{PSL}(2,\mathbb{C})/\mathrm{PSU}(2)}}$$

$$= \frac{1}{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}\det(\frac{\partial\chi_i}{\partial b_j}).$$

The situation for the disk is very similar to that of the sphere, except that we only have a $\mathrm{PSL}(2,\mathbb{R})$ subgroup of the conformal symmetry. The analogous Fadeev-Popov procedure is

$$\frac{\mathrm{d}(\text{disk zero modes})}{\mathrm{d}(\mathrm{PSL}(2,\mathbb{R}))} = \frac{2^{3/2}\mathrm{d}\chi_1\mathrm{d}\chi_2}{\underbrace{\mathrm{d}s_1}_{\mathrm{U}(1)}\ \underbrace{\mathrm{d}b_1\mathrm{d}b_2}_{\mathrm{PSL}(2,\mathbb{R})/\mathrm{U}(1)}}$$

$$= \frac{2^{3/2}}{\mathrm{d}s_1}\det(\frac{\partial\chi_i}{\partial b_j}).$$

Note that we inserted an important factor of $2^{3/2}$. This factor will be explained below.

Let's now work out the details explicitly. We write the reference sphere or hemisphere in stereographic coordinates

$$\mathrm{d}\hat{s}^2 = \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{(1+z\bar{z})^2}. \qquad (1.25)$$

The infinitesimal PSL$(2,\mathbb{C})$ or PSL$(2,\mathbb{R})$ transformations correspond to the following set of six holomorphic vector fields (c ghost zero modes)

a	$C^z_{0,a} = \delta_a z$	$C^{\bar{z}}_{0,a} = \delta_a \bar{z}$	coordinate
1	$\mathrm{i}z$	$-\mathrm{i}\bar{z}$	s_1
2	$\frac{1}{2}(1-z^2)$	$\frac{1}{2}(1-\bar{z}^2)$	b_1
3	$\frac{\mathrm{i}}{2}(1+z^2)$	$-\frac{\mathrm{i}}{2}(1+\bar{z}^2)$	b_2
4	z	$\bar{z}$	b_3
5	$\frac{\mathrm{i}}{2}(1-z^2)$	$-\frac{\mathrm{i}}{2}(1-\bar{z}^2)$	s_2
6	$\frac{1}{2}(1+z^2)$	$\frac{1}{2}(1+\bar{z}^2)$	s_3

(1.26)

The first three of these vector fields preserve the hemisphere $|z| \leq 1$, and these correspond to the PSL$(2,\mathbb{R})$ subgroup of PSL$(2,\mathbb{C})$. The last three make sense only on the full sphere.

One way to compute the group measure in the s_i and b_i coordinates is to take the square root of the determinant of the field-space inner product of these vector fields

$$M_{ab} = \frac{3}{8\pi} \int \sqrt{\hat{g}} C^{\alpha}_{0,a} C^{\beta}_{0,b} \hat{g}_{\alpha\beta}. \qquad (1.27)$$

The constant $3/8\pi$ out front will cancel out in the ratio sphere/(disk)^2, and we chose it to so that the answer is simply that M is the 6×6 identity matrix for the sphere, and one-half of the 3×3 identity matrix for the disk. So the measure is one for the sphere, and $2^{-3/2}$ for the disk.

It remains to compute the determinants of $\partial\chi_i/\partial q_j$. Conformal transformations are defined to act on the Liouville field in such a way that the physical metric remains invariant. So, under a general

$$z \to \widetilde{z}(z) \tag{1.28}$$

we require that

$$e^{2\widetilde{\sigma}(\widetilde{z},\bar{\widetilde{z}})}\frac{\mathrm{d}\widetilde{z}\mathrm{d}\bar{\widetilde{z}}}{(1+\widetilde{z}\bar{\widetilde{z}})^2} = e^{2\sigma(z,\bar{z})}\frac{\mathrm{d}z\mathrm{d}\bar{z}}{(1+z\bar{z})^2}. \tag{1.29}$$

One finds that for the transformations (1.26), the corresponding perturbations to the classical solution σ_{cl} are

$$\text{sphere:} \quad \delta_a\sigma = \left\{0, \frac{z+\bar{z}}{1+z\bar{z}}, -\mathrm{i}\frac{z-\bar{z}}{1+z\bar{z}}, \frac{-1+z\bar{z}}{1+z\bar{z}}, 0, 0\right\}$$

$$\text{disk:} \quad \delta_a\sigma = \frac{1}{2}\left\{0, z+\bar{z}, -\mathrm{i}(z-\bar{z})\right\}.$$

We see that the $a = 1, 5, 6$ directions are the PSU(2) symmetry directions that stabilize the classical solution, justifying the labeling in (1.26).

The nonzero $\delta_a\sigma$ functions correspond precisely to the zero modes of the one-loop determinants, but we have to fix their normalization. The χ_i coordinates are the coefficients of *normalized* zero modes. To see the discrepancy, we can evaluate the matrix $m_{ab} = (\delta_a\sigma, \delta_b\sigma)$ where the inner product is defined in (1.12). One finds

$$\text{sphere:} \quad m = \frac{1}{3}\text{diag}(0,1,1,1,0,0) \tag{1.30}$$

$$\text{disk:} \quad m = \frac{\log(4)-1}{4}\text{diag}(0,1,1). \tag{1.31}$$

The determinants $\det(\partial\chi_i/\partial q_j)$ are just the square roots of the determinants of the nonzero submatrices here,

$$\text{sphere:} \quad \det(\frac{\partial\chi_i}{\partial q_j}) = \frac{1}{3^{3/2}} \tag{1.32}$$

$$\text{disk:} \quad \det(\frac{\partial\chi_i}{\partial q_j}) = \frac{\log(4)-1}{4}. \tag{1.33}$$

So the gauge-fixing factors should be in the two cases

$$\frac{1}{\mathrm{vol}(\mathrm{PSL}(2,\mathbb{C}))} = \frac{1}{\mathrm{vol}(\mathrm{PSU}(2))}\frac{1}{3^{3/2}}\delta(\chi_1)\delta(\chi_2)\delta(\chi_3) \tag{1.34}$$

$$\frac{1}{\mathrm{vol}(\mathrm{PSL}(2,\mathbb{R}))} = \frac{2^{3/2}}{\mathrm{vol}(\mathrm{U}(1))}\frac{\log(4)-1}{4}\delta(\chi_1)\delta(\chi_2). \tag{1.35}$$

We normalized the original transformations (1.26) so that with a unit measure, a full rotation has length 2π. For the case of $U(1)$, this means simply $\mathrm{vol}(\mathrm{U}(1)) = 2\pi$. For the case of $\mathrm{PSU}(2)$, we can use the fact $\mathrm{vol}(\mathrm{PSU}(2)) = 2\pi\,\mathrm{vol}(S^2) = 8\pi^2$.

1.4 Putting the pieces together

$$\frac{Z_{\text{sphere}}}{\text{vol}(\text{PSL}(2,\mathbb{C}))} = \left(\frac{\mathrm{i}e^{\frac{1}{b^2}}}{2\sin(\frac{\pi}{b^2})}(4\pi\mu)^{\frac{1}{b^2}+1}\right)$$

$$\left(-\mathrm{i}\frac{\sqrt{\pi}\,bC}{\sqrt{2}}\cdot C^3\cdot\frac{(\sqrt{\pi}bC)^{-\frac{11}{3}}}{e^{\frac{1}{2}\cdot 2.32713}}\right)$$

$$\left(\frac{1}{8\pi^2 3^{3/2}}\right)$$

$$\frac{Z_{\text{disk}}}{\text{vol}(\text{PSL}(2,\mathbb{R}))} = \left(\frac{-\mathrm{i}e^{\frac{1}{2b^2}(1+2\pi\mathrm{i})}}{e\sin(\frac{\pi}{b^2})}\left(\frac{\sqrt{4\pi\mu}}{\alpha}\right)^{\frac{1}{b^2}+1}\right)$$

$$\left(\frac{-\mathrm{i}\sqrt{\pi}\,bC}{\sqrt{1.51095}}\cdot C^2\cdot\frac{(\sqrt{\pi}bC)^{-\frac{17}{6}}}{e^{\frac{1}{2}\cdot 1.14858}}\right)$$

$$\left(\frac{2^{\frac{3}{2}}(\log(4)-1)}{2\pi\cdot 4}\right)$$

The invariant ratio is

$$\frac{Z_{\rm sphere}}{\mathrm{vol}(\mathrm{PSL}(2,\mathbb{C}))} \cdot \left(\frac{\mathrm{vol}(\mathrm{PSL}(2,\mathbb{R}))}{Z_{\rm disk}} \right)^2 = 8.889\, e^{-\frac{2\pi \mathrm{i}}{b^2}} b \sin(\frac{\pi}{b^2})\, \alpha^{2+\frac{2}{b^2}}.$$

There is one final step. To compute the partition functions of the minimal string, we set $b = \sqrt{2/p}$ where p is an odd integer (and which must be large for our semiclassical approximation to be valid) and multiply by the partition function of the matter sector, which is the $(2, p)$ minimal model.

Using the formula (here $S_{(1,1),(1,1)}$ is an element of the modular S-matrix relating the identity characters in the two channels)

$$\frac{(Z^{\text{minimal model}}_{\text{disk }(1,1)})^2}{Z^{\text{minimal model}}_{\text{sphere}}} = S_{(1,1),(1,1)}$$

$$= -\frac{2}{\sqrt{p}} \sin(\frac{\pi p}{2}) \sin(\frac{2\pi}{p}),$$

and approximating $\sin(\frac{2\pi}{p}) = \frac{2\pi}{p}$, we find that for large p and small α

$$\frac{\mathcal{Z}_{\text{sphere}}}{(\mathcal{Z}_{\text{disk}})^2} = 1.000\, p\, \alpha^{p+2}. \qquad (1.36)$$

The sphere partition function is related to the leading term $L^2\mathcal{F}_0$ in the logarithm of the full matrix partition function:

$$\log(\mathfrak{Z}) = L^2\mathcal{F}_0 + \mathcal{F}_1 + L^{-2}\mathcal{F}_2 + \dots . \qquad (2.1)$$

One can get this term by simply evaluating the action I on the stationary configuration ρ_0:

$$\mathcal{F}_0 = -I[\rho_0]. \qquad (2.2)$$

The FZZT disk partition function is given by a similar leading term $L\mathcal{G}_0$ in the expectation value

$$\langle \mathrm{Tr}\log(H - x)\rangle = L\mathcal{G}_0(x) + L^{-1}\mathcal{G}_1(x) + \dots$$

Again, this is given simply in terms of the stationary configuration ρ_0:

$$\mathcal{G}_0(x) = \int \mathrm{d}\lambda\, \rho_0(\lambda)\log(\lambda - x). \qquad (2.3)$$

Let $p = 2m-1$. We engineer the a one-parameter family of polynomial potentials, with parameter ϵ and with all coefficients of the polynomial being analytic in ϵ^2, so that near the left edge of the spectrum, we have the density of states

$$\rho_0(\lambda) = \frac{2^{m+\frac{1}{2}}\epsilon^{m-\frac{1}{2}}}{\pi\binom{2m-1}{m-1}} \sinh\left[\frac{2m-1}{2}\operatorname{arccosh}(1+2E)\right] + \mathcal{O}(\epsilon^{m+\frac{1}{2}})$$

Using Mathematica, we found the first few cases:

$$2\mathcal{F}_0^{(2,3)} = -\log 2 - \frac{25}{24} - \frac{\epsilon^2}{3} + \frac{\epsilon^4}{4} \boxed{-\frac{4\epsilon^5}{15}} + \dots \tag{2.4}$$

$$2\mathcal{F}_0^{(2,5)} = -\log 2 - \frac{49}{40} - \frac{\epsilon^2}{5} + \frac{\epsilon^4}{24} \boxed{-\frac{4\epsilon^7}{105}} + \dots \tag{2.5}$$

Here we have boxed the first nonanalytic term in each case. This is the universal part of the answer. All of the other terms in the ϵ expansion are "nonuniversal garbage," depending on specific decisions we made in constructing the double-scaled theory.

In fact there is a simple general answer for arbitrary $p = 2m - 1$:

$$2\mathcal{F}_0^{(2,p)} = (\text{analytic in } \epsilon^2) - \frac{2^{p-1}p}{(p^2-4)\binom{p}{\frac{p-1}{2}}^2}\epsilon^{p+2} + \mathcal{O}(\epsilon^{p+3}).$$

To derive this in an efficient way, one can use the genus-zero string equation. See paper for more details.

Recall that

$$\mathcal{G}_0(x) = \int_{-a}^{a} \mathrm{d}\lambda \rho_0(\lambda) \log(\lambda - x). \qquad (2.6)$$

To define a function appropriate for the double-scaled limit, we consider this function at an argument x that is close to the lower endpoint. For example, in the case of the $(2,3)$ model, we have the explicit formula

$$\begin{aligned}\mathcal{G}_0^{(2,3)}(-a+\epsilon E) = \frac{7}{12} - \frac{2(1+2E)}{3}\epsilon \\ + \left(\frac{2}{3} + \frac{(1+2E)^2}{2}\right)\epsilon^2 \\ + \boxed{\mathrm{i}\frac{2^{7/2}}{15}E^{3/2}(5+4E)\epsilon^{5/2}} + \mathcal{O}(\epsilon^3).\end{aligned}$$

We have boxed the universal term, which will be compared to the Liouville computations below.

At first, this might seem puzzling, because the "nonuniversal garbage" contains a term ϵ^1, which would seem to be a nonanalytic function of $\mu \sim \epsilon^2$. However, from the Liouville perspective, precisely the combination $(1+2E)\epsilon$ is proportional to μ_B, the boundary cosmological constant, and we should expect nonuniversal analytic terms in both μ and μ_B.

In the Liouville computation, the disk path integral was pure imaginary, which suggests that the first nonanalytic term will be pure imaginary. This is true of $(2,3)$, and also true for the $(2,5)$ and $(2,7)$ cases, which we checked explicitly. We don't have a general proof of this from the matrix side, although we suspect it is possible to show this.

$$\begin{aligned}\operatorname{Im}\mathcal{G}_0(-a+\epsilon E) &= \operatorname{Im}\int_{-a}^{a} \mathrm{d}\lambda \rho_0(\lambda)\log(\lambda+a-\epsilon E) \\ &= \pi\int_{-a}^{-a+\epsilon E} \mathrm{d}\lambda\rho_0(\lambda).\end{aligned}$$

This only depends on the density of states near the edge, where it is constrained by the double-scaled limit. Putting in the sinh arccosh formula for the density and taking high energies $E \gg 1$, this is

$$\operatorname{Im}\mathcal{G}_0(-a+\epsilon E) \to \frac{(2\epsilon)^{\frac{p}{2}+1}}{4\binom{p}{\frac{p-1}{2}}}\frac{(4E)^{\frac{p}{2}+1}}{p+2}. \qquad (2.7)$$

2.4 Comparing the ratio to the Liouville answer

As a final step, we need to relate the α parameter of the Liouville theory to the matrix energy E.

We can compare $Z'_{\text{disk}}(\mu_B)$ with $\mathcal{G}'_0(x)$, which is proportional to $\rho_0(E)$. The leading α dependence of $Z'_{\text{disk}}(\mu_B)$ is proportional to

$$e^{-\frac{1}{b^2}\log(\alpha)} = e^{\frac{p}{2}\log(\frac{1}{\alpha})}. \tag{2.8}$$

On the other hand, for large p, the density $\rho_0(E)$ is proportional to

$$e^{\frac{p}{2}\text{arccosh}(1+2E)}. \tag{2.9}$$

Comparing the two, we conclude that

$$\log\frac{1}{\alpha} = \text{arccosh}(1+2E). \tag{2.10}$$

Using the relation between α and μ_B, this implies $-\sqrt{\frac{\pi}{\mu}}\mu_B = 1 + 2E$, which justifies a statement made above that $(1+2E)\epsilon \propto \mu_B$. At high energies,

$$\frac{1}{\alpha} \approx 4E. \tag{2.11}$$

$$\frac{\mathcal{F}_0^{(2,p)}|_{\text{universal}}}{(\mathcal{G}_0^{(2,p)}|_{\text{universal}})^2} = \frac{p+2}{p-2}\, p\, \alpha^{p+2}, \quad \alpha \ll 1. \tag{2.12}$$

This exact answer agrees with (1.36) at large p.

3 Some vague comments

3.1 Action of supersymmetric D-branes

The volume of OSp(1|2) is finite, and can be used to give a path integral derivation of the D-brane tension in flat 10-d space.
`Eberhardt-Pal,`
`Mahajan-Stanford-Yan (unpublished).`

The recursion relation between the tension of a Dp brane and a D$(p-1)$ brane can be obtained as the ratio of the disk partition function of a scalar with D boundary conditions with the disk partition function of a scalar with N boundary conditions.

3.2 Tseytlin's papers

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THE PARTITION FUNCTION OF THE SIGMA MODEL OF THE STRING ON A COMPACT TWO-SPACE

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We develop a manifestly covariant procedure of computation of the partition function Z of a general renormalizable 2D sigma model defined on a compact two-space. Special attention is paid to a proper separation of the path integral into the integrals over the constant and nonconstant parts of the coordinate. The important role of the path integral measure in establishing the covariance and renormalizability properties of Z is stressed. We explicitly check the validity of the representation of the closed string effective action as the partial derivative of Z over the logarithm of the cutoff. We also comment on the status of the "c-theorem" and suggest a resolution of a problem in the renormalization of the closed string loop corrections corresponding to the disc topology.

The partition function of a CFT satisfies

$$\frac{Z(R_1)}{Z(R_0)} = \left(\frac{R_1}{R_0}\right)^{\frac{c}{3}} \tag{3.1}$$

For a sigma model on the string worldsheet, separate the ws fields into zero modes and do the path integral over the non-zero modes, with sphere worldsheet topology. The resulting expression, which is a function of the zero-modes x^μ, is interpreted as a "central-charge functional"

$$c\,(G_{\mu\nu}(x), \ldots) \tag{3.2}$$

The claim of Tseytlin is that this function is the action-density that appears in the spacetime effective action in string theory.

Naive check: The bulk term in the spacetime effective action vanishes on-shell. Being on-shell is the same as having no conformal anomaly, or $c = 0$.

3.3 Dilaton zero mode

The dilaton zero mode is a universal operator. Its coefficient in the action is the string coupling. So we can try to compute its three point function on the sphere.

$$\frac{\partial^3}{\partial g_s^3} Z_{\mathrm{sphere}} = \langle \Phi_0 \Phi_0 \Phi_0 \rangle_{\mathrm{sphere}} \tag{3.3}$$

$$\frac{\partial}{\partial g_s} Z_{\mathrm{disk}} = \langle \Phi_0 \rangle_{\mathrm{disk}} \tag{3.4}$$

First compute the RHS using some method, and then integrate it thrice.

What is the simplest background in which this can be done?

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